MATEMATIČKI VESNIK
МАТЕМАТИЧКИ ВЕСНИК
Corrected proof
Available online 21.08.2024

research paper
оригинални научни рад
DOI: 10.57016/MV-EhnQ1530

# DISTANCE BETWEEN EDGES IN STRONG DOUBLE GRAPHS AND RELATED GRAPH INVARIANTS

**Mahdieh Azari**

**Abstract**. The distance between two edges in a simple connected graph can be described as the distance between the corresponding vertices in its line graph. In this paper, we determine the distance between edges in strong double graphs and apply our results to compute some edge distance-related invariants for this family of graphs.

## 1. Introduction

All graphs considered in this paper are finite, simple and connected. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of a vertex $u \in V(G)$ is the number of edges incident to $u$. Two distinct edges $e = uv$ and $f = zt$ of $G$ are considered adjacent if they have a vertex in common. The line graph $L(G)$ of $G$ is the graph whose vertices correspond to the edges of $G$, with two vertices being adjacent if and only if the corresponding edges are adjacent in $G$. The degree $d_G(e)$ of an edge $e = uv \in E(G)$ is the degree of the corresponding vertex $e$ in the line graph of $G$, which is equal to $d_G(e) = d_G(u) + d_G(v) - 2$. We denote with $N_G(e)$ the set of all edges adjacent to $e \in E(G)$ and with $\delta_G(e)$ the sum of the degrees of all edges adjacent to $e$, i.e. $\delta_G(e) = \sum_{f \in N_G(e)} d_G(f)$.

The distance $d_G(u, v)$ between the vertices $u, v \in V(G)$ is the length of any shortest path in $G$ that contains $u$ and $v$. The transmission (also called status) $D_G(u)$ of a vertex $u \in V(G)$ is the sum of the distances between $u$ and all other vertices $v$ of $G$, i.e. $D_G(u) = \sum_{v \in V(G)} d_G(u, v)$. We use $D_G^{(2)}(u)$ to denote the sum of the squares of the distances between a vertex $u \in V(G)$ and all other vertices $v$ of $G$, i.e. $D_G^{(2)}(u) = \sum_{v \in V(G)} d_G(u, v)^2$.

The distance $d_G(e, f)$ between the edges $e = uv$ and $f = zt$ of $G$ was described in [7, 15, 18] as the distance between the corresponding vertices $e$ and $f$ in the line

*2020 Mathematics Subject Classification*: 05C12, 05C76
*Keywords and phrases*: Distance between edges in graph; graph invariants; strong double graphs.

graph of $G$. If $e$ and $f$ are distinct, then it was proved in [15] that,

$$d_G(e,f) = \min\{d_G(u,z), d_G(u,t), d_G(v,z), d_G(v,t)\} + 1.$$

We denote with $D_G(e)$ the transmission of the edge $e \in E(G)$, which is the sum of the distances between $e$ and all other edges $f$ of $G$, i.e. $D_G(e) = \sum_{f \in E(G)} d_G(e,f)$, and with $D_G^{(2)}(e)$ the sum of the squares of the distances between an edge $e \in E(G)$ and all other edges $f$ of $G$, i.e. $D_G^{(2)}(e) = \sum_{f \in E(G)} d_G(e,f)^2$.

The distance between a vertex $u$ and an edge $e = xy$ of $G$ was proposed in [6,18] as $d_G(u,e) = d_G(e,u) = \min\{d_G(u,x), d_G(u,y)\}$.

We denote with $D'_G(u)$ the sum of the distances between a vertex $u \in V(G)$ and all edges $e$ of $G$, i.e. $D'_G(u) = \sum_{e \in E(G)} d_G(u,e)$, and with $D'^{(2)}_G(u)$ the sum of the squares of the distances between a vertex $u \in V(G)$ and all edges $e$ of $G$, i.e. $D'^{(2)}_G(u) = \sum_{e \in E(G)} d_G(u,e)^2$. We denote with $D'_G(e)$ the sum of the distances between an edge $e \in E(G)$ and all vertices $u$ of $G$, i.e. $D'_G(e) = \sum_{u \in V(G)} d_G(u,e)$ and with $D'^{(2)}_G(e)$ the sum of the squares of the distances between an edge $e \in E(G)$ and all vertices $v$ of $G$, i.e, $D'^{(2)}_G(e) = \sum_{u \in V(G)} d_G(u,e)^2$.

In the field of chemical graph theory, a topological index is a numerical parameter of a graph that characterizes its topology and is usually graph invariant. Topological indices are used in the development of quantitative structure-activity relationships (QSARs), in which the biological activity or other properties of molecules are correlated with their chemical structure.

The Wiener index, introduced by Wiener [29] in 1947, was the first topological index to be recognized in chemical graph theory. The Wiener index of a graph $G$ was formulated by

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{u \in V(G)} D_G(u).$$

The edge-Wiener index of a graph $G$ was introduced by Iranmanesh et al. [15] as

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d_G(e,f) = \frac{1}{2} \sum_{e \in E(G)} D_G(e).$$

The edge-Wiener index of a graph can also be introduced as the Wiener index of its line graph [7]. Another definition for the edge-Wiener index was given in [18].

The vertex-edge Wiener index of a graph $G$ was introduced in [6,18] as

$$W_{ve}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} d_G(u,e) = \sum_{u \in V(G)} D'_G(u) = \sum_{e \in E(G)} D'_G(e).$$

The hyper-Wiener index of acyclic graphs was introduced by Randić [24] in 1993. Then Klein et al. [19] generalized Randić's definition for all connected graphs in 1995. The hyper-Wiener index of a graph $G$ was formulated as

$$WW(G) = \frac{1}{2}\Big(W(G) + W^{(2)}(G)\Big),$$

where $$W^{(2)}(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)^2 = \frac{1}{2}\sum_{u\in V(G)} D_G^{(2)}(u).$$

The edge hyper-Wiener index of a graph $G$ was defined by Iranmanesh et al. [17] as

$$WW_e(G) = \frac{1}{2}\Big(W_e(G) + W_e^{(2)}(G)\Big),$$

where $$W_e^{(2)}(G) = \sum_{\{e,f\}\subseteq E(G)} d_G(e,f)^2 = \frac{1}{2}\sum_{e\in E(G)} D_G^{(2)}(e).$$

The vertex-edge hyper-Wiener index of a graph $G$ was proposed in [5] as

$$WW_{ve}(G) = W_{ve}(G) + W_{ve}^{(2)}(G),$$

where $$W_{ve}^{(2)}(G) = \sum_{u\in V(G)}\sum_{e\in E(G)} d_G(u,e)^2 = \sum_{u\in V(G)} D_G'^{(2)}(u) = \sum_{e\in E(G)} D_G'^{(2)}(e).$$

The degree distance was introduced by Dobrynin and Kochetova [8] and at the same time by Gutman [10] as a weighted version of the Wiener index. It was formulated for a graph $G$ as

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} \big(d_G(u) + d_G(v)\big) d_G(u,v) = \sum_{u\in V(G)} d_G(u) D_G(u).$$

The edge-degree distance of a graph $G$ was defined by Iranmanesh et al. [16] as

$$DD_e(G) = \sum_{\{e,f\}\subseteq E(G)} \big(d_G(e) + d_G(f)\big) d_G(e,f) = \sum_{e\in E(G)} d_G(e) D_G(e).$$

We define the vertex-edge degree distance of a graph $G$ as

$$DD_{ve}(G) = \sum_{u\in V(G)} d_G(u) D'_G(u),$$

and the edge-vertex degree distance of $G$ as

$$DD_{ev}(G) = \sum_{e\in E(G)} d_G(e) D'_G(e).$$

The Gutman index was introduced in 1994 by Gutman [10] as a kind of vertex-valency-weighted sum of the distances between all pairs of vertices in a graph. The Gutman index of a graph $G$ was formulated as

$$Gut(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u) d_G(v) d_G(u,v) = \frac{1}{2}\sum_{u\in V(G)} d_G(u) S_G(u),$$

where $$S_G(u) = \sum_{v\in V(G)} d_G(v) d_G(u,v).$$

The edge-Gutman index of a graph $G$ was defined by Iranmanesh et al. [16] as

$$Gut_e(G) = \sum_{\{e,f\}\subseteq E(G)} d_G(e) d_G(f) d_G(e,f) = \frac{1}{2}\sum_{e\in E(G)} d_G(e) S_G(e),$$

where $$S_G(e) = \sum_{f \in E(G)} d_G(f) d_G(e,f).$$

We define the vertex-edge-Gutman index of a graph $G$ as

$$Gut_{ve}(G) = \sum_{u \in V(G)} \sum_{e \in E(G)} d_G(u) d_G(e) d_G(u,e).$$

The vertex-edge-Gutman index can also be expressed as

$$Gut_{ve}(G) = \sum_{u \in V(G)} d_G(u) S'_G(u) = \sum_{e \in E(G)} d_G(e) S'_G(e),$$

where $$S'_G(u) = \sum_{e \in E(G)} d_G(e) d_G(u,e), \quad S'_G(e) = \sum_{u \in V(G)} d_G(u) d_G(u,e).$$

The first Zagreb index of a graph $G$ was introduced by Gutman and Trinajstić [11] in 1972 and the second Zagreb index was proposed by Gutman et al. [12] in 1975 as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} \big(d_G(u) + d_G(v)\big), \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The forgotten topological index [9, 11] was expressed for a graph $G$ as

$$F(G) = \sum_{u \in V(G)} d_G(u)^3 = \sum_{uv \in E(G)} \big(d_G(u)^2 + d_G(v)^2\big).$$

It can be easily verified that ,

$$\begin{aligned} \sum_{e \in E(G)} \delta_G(e) = M_1(L(G)) &= F(G) + 2M_2(G) - 4M_1(G) + 4|E(G)|, \\ \sum_{e \in E(G)} d_G(e) \delta_G(e) &= 2M_2(L(G)). \end{aligned} \tag{1}$$

Many graphs are obtained from simpler graphs by the use of graph transformations and, as a consequence, the properties of the resulting graphs are strongly related to the properties of their parent graphs. For this reason, the study of graph transformations is an important research topic in graph theory. The present work deals with the study of an interesting family of graph transformations, the so-called strong double graphs. Some vertex-degree-based invariants of strong double graphs were computed in [13, 22, 23]. Sardar et al. [25, 26] determined some topological indices of the strong double graph of silicon carbide $Si_2C_3 - I[p,q]$ and the circumcoronene series of benzenoid. Ali et al. [2] considered vertex-degree-based topological indices of the strong double graph of the Dutch windmill graph. Akhter et al. [1] investigated the relationship between the vertex version of the Wiener, hyper-Wiener, degree distance and Gutman indices of the strong double graph with the corresponding invariants of its parent graph. Here we follow this process to study such relationship for the edge versions of these graph invariants. Readers interested in more information on the mathematical properties, theory, and applications of edge distance-based topological indices can consult [3, 4, 14, 21, 27, 28, 30], and the references therein.

## 2. Definitions and preliminaries

We start this section with definition of the strong double graph.

Definition 2.1. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The strong double graph $\mathcal{SD}[G]$ of $G$ is obtained by taking two distinct copies $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ of $G$ by preserving the primary edge set of each copy and joining each vertex in one copy with the closed neighborhood of the corresponding vertex in another copy (see [20]).

From Definition 2.1, $\mathcal{SD}[G]$ has $2n$ vertices and $n + 4m$ edges, where $m$ is the number of edges of $G$. The degree of any vertex and the distance between two arbitrary vertices in strong double graph were computed in [1].

Lemma 2.2 ([1]). *The degree of each vertex in $\mathcal{SD}[G]$ is given by*

$$d_{\mathcal{SD}[G]}(x_i) = d_{\mathcal{SD}[G]}(y_i) = 2d_G(v_i) + 1.$$

Lemma 2.3 ([1]). *The distance between each pair of vertices in $\mathcal{SD}[G]$ is given by*

$$d_{\mathcal{SD}[G]}(x_i, x_j) = d_{\mathcal{SD}[G]}(y_i, y_j) = d_G(v_i, v_j),\ d_{\mathcal{SD}[G]}(x_i, y_j) = \begin{cases} d_G(v_i, v_j), & i \neq j, \\ 1, & i = j. \end{cases}$$

In the following lemmas, we compute the degree of an edge and the distance between edges in strong double graph. The results follow from Lemmas 2.2 and 2.3 and the proofs are omitted.

Lemma 2.4. *For $1 \leq i \leq n$, $d_{\mathcal{SD}[G]}(x_i y_i) = 4d_G(v_i)$, and for each edge $v_i v_j \in E(G)$,*

$$d_{\mathcal{SD}[G]}(x_i x_j) = d_{\mathcal{SD}[G]}(y_i y_j) = d_{\mathcal{SD}[G]}(x_i y_j) = d_{\mathcal{SD}[G]}(x_j y_i) = 2d_G(v_i v_j) + 4.$$

Lemma 2.5. *For each pair of edges $v_i v_j, v_r v_s \in E(G)$, we have*

$$d_{\mathcal{SD}[G]}(x_i x_j, x_r x_s) = d_{\mathcal{SD}[G]}(y_i y_j, y_r y_s) = d_G(v_i v_j, v_r v_s);$$

$$d_{\mathcal{SD}[G]}(x_i x_j, x_r y_r) = d_{\mathcal{SD}[G]}(y_i y_j, x_r y_r) = d_{\mathcal{SD}[G]}(x_i y_j, x_r y_r) = d_G(v_i v_j, v_r) + 1;$$

$$d_{\mathcal{SD}[G]}(x_i x_j, x_r y_s) = d_{\mathcal{SD}[G]}(y_i y_j, x_s y_r) = \begin{cases} d_G(v_i v_j, v_r v_s) + 1, & s \in \{i, j\}, \\ d_G(v_i v_j, v_r v_s), & s \notin \{i, j\}; \end{cases}$$

$$d_{\mathcal{SD}[G]}(x_i x_j, y_r y_s) = \begin{cases} d_G(v_i v_j, v_r v_s) + 2, & \{r, s\} = \{i, j\}, \\ d_G(v_i v_j, v_r v_s) + 1, & r \in \{i, j\} \text{ or } s \in \{i, j\}, \{r, s\} \neq \{i, j\}, \\ d_G(v_i v_j, v_r v_s) & r, s \notin \{i, j\}; \end{cases}$$

$$d_{\mathcal{SD}[G]}(x_i y_j, x_r y_s) = \{ \begin{cases} d_G(v_i v_j, v_r v_s) + 2, & r = j, s = i, \\ d_G(v_i v_j, v_r v_s) + 1, & r = j, s \neq i \text{ or } s = i, r \neq j, \\ d_G(v_i v_j, v_r v_s), & r = i, s = j \text{ or } r, s \notin \{i, j\}; \end{cases}$$

$$d_{\mathcal{SD}[G]}(x_i y_i, x_r y_r) = \begin{cases} 0, & r = i, \\ d_G(v_i, v_r) + 1, & r \neq i. \end{cases}$$

## 3. Main results

In this section, we study the edge version of the Wiener, hyper-Wiener, degree distance, and Gutman indices for strong double graph. Throughout this section, let $G$ be a graph of size $m$ and vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let $V(\mathcal{SD}[G]) = X \cup Y$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are two distinct copies of $V(G)$.

Theorem 3.1. *The edge-Wiener index of $\mathcal{SD}[G]$ is given by*

$$W_e(\mathcal{SD}[G]) = 16W_e(G) + 4W_{ve}(G) + W(G) + 4M_1(G) + \binom{n}{2} + 4nm. \qquad (2)$$

*Proof.* Corresponding to each $v_iv_j \in E(G)$, there exist four edges $x_ix_j, y_iy_j, x_iy_j, x_jy_i \in E(\mathcal{SD}[G])$. Then

$$\begin{aligned} D_{\mathcal{SD}[G]}(x_ix_j) = & \sum_{x_rx_s \in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_ix_j, x_rx_s) + \Big[ d_{\mathcal{SD}[G]}(x_ix_j, y_iy_j) \\ & + \sum_{\substack{y_iy_s \in E(\mathcal{SD}[G]):\\ s\neq j}} d_{\mathcal{SD}[G]}(x_ix_j, y_iy_s) + \sum_{\substack{y_ry_j \in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_ix_j, y_ry_j) \\ & + \sum_{\substack{y_ry_s \in E(\mathcal{SD}[G]):\\ r,s\neq i,j}} d_{\mathcal{SD}[G]}(x_ix_j, y_ry_s) \Big] + \Big[ \sum_{\substack{x_ry_i \in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_i) \\ & + \sum_{\substack{x_ry_j \in E(\mathcal{SD}[G]):\\ r\neq j}} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_j) + \sum_{\substack{x_ry_s \in E(\mathcal{SD}[G]):\\ s\neq r,i,j}} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_s) \Big] \\ & + \sum_{r=1}^{n} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_r). \end{aligned}$$

Now from Lemma 2.5, we get

$$\begin{aligned} D_{\mathcal{SD}[G]}(x_ix_j) = & \sum_{v_rv_s \in E(G)} d_G(v_iv_j, v_rv_s) + \Big[ \big( d_G(v_iv_j, v_iv_j) + 2 \big) \\ & + \sum_{\substack{v_iv_s \in E(G):\\ s\neq j}} \big( d_G(v_iv_j, v_iv_s) + 1 \big) + \sum_{\substack{v_rv_j \in E(G):\\ r\neq i}} \big( d_G(v_iv_j, v_rv_j) + 1 \big) \\ & + \sum_{\substack{v_rv_s \in E(G):\\ r,s\neq i,j}} d_G(v_iv_j, v_rv_s) \Big] + \Big[ \sum_{v_rv_i \in E(G)} \big( d_G(v_iv_j, v_rv_i) + 1 \big) \\ & + \sum_{v_rv_j \in E(G)} \big( d_G(v_iv_j, v_rv_j) + 1 \big) + \sum_{\substack{v_rv_s \in E(G):\\ s\neq i,j}} d_G(v_iv_j, v_rv_s) \Big] \\ & + \sum_{r=1}^{n} \big( d_G(v_iv_j, v_r) + 1 \big) \end{aligned}$$

$$
\begin{aligned}
&= D_G(v_iv_j) + \big(2 + D_G(v_iv_j) + d_G(v_i) - 1 + d_G(v_j) - 1\big) \\
&\quad + \big(2D_G(v_iv_j) + d_G(v_i) + d_G(v_j)\big) + \big(D'_G(v_iv_j) + n\big) \\
&= 4D_G(v_iv_j) + D'_G(v_iv_j) + 2\big(d_G(v_i) + d_G(v_j)\big) + n.
\end{aligned}
$$

By symmetry, for each edge $v_iv_j \in E(G)$, we have

$$
\begin{aligned}
D_{\mathcal{SD}[G]}(y_iy_j) &= D_{\mathcal{SD}[G]}(x_iy_j) = D_{\mathcal{SD}[G]}(x_jy_i) = D_{\mathcal{SD}[G]}(x_ix_j) \\
&= 4D_G(v_iv_j) + D'_G(v_iv_j) + 2\big(d_G(v_i) + d_G(v_j)\big) + n.
\end{aligned} \tag{3}
$$

Similarly for $1 \leq i \leq n$, we have

$$
\begin{aligned}
D_{\mathcal{SD}[G]}(x_iy_i) = &\sum_{x_rx_s \in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_iy_i, x_rx_s) + \sum_{y_ry_s \in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_iy_i, y_ry_s) \\
&+ \sum_{\substack{x_ry_s \in E(\mathcal{SD}[G]): \\ r \neq s}} d_{\mathcal{SD}[G]}(x_iy_i, x_ry_s) + \sum_{\substack{x_ry_r \in E(\mathcal{SD}[G]): \\ r \neq i}} d_{\mathcal{SD}[G]}(x_iy_i, x_ry_r).
\end{aligned}
$$

Now from Lemma 2.5, we get

$$
\begin{aligned}
D_{\mathcal{SD}[G]}(x_iy_i) = &\sum_{v_rv_s \in E(G)} \big(d_G(v_rv_s, v_i) + 1\big) + \sum_{v_rv_s \in E(G)} \big(d_G(v_rv_s, v_i) + 1\big) \\
&+ 2\sum_{v_rv_s \in E(G)} \big(d_G(v_rv_s, v_i) + 1\big) + \sum_{\substack{r=1, \\ r \neq i}}^{n} \big(d_G(v_i, v_r) + 1\big) \\
= &\, 4\big(D'_G(v_i) + m\big) + D_G(v_i) + n - 1,
\end{aligned}
$$

from which

$$
D_{\mathcal{SD}[G]}(x_iy_i) = 4D'_G(v_i) + D_G(v_i) + n - 1 + 4m. \tag{4}
$$

Now from the definition of the edge-Wiener index and (3) and (4), we obtain

$$
\begin{aligned}
W_e(\mathcal{SD}[G]) &= \frac{1}{2}\Big(4\sum_{x_ix_j \in E(G)} D_{\mathcal{SD}[G]}(x_ix_j) + \sum_{i=1}^{n} D_{\mathcal{SD}[G]}(x_iy_i)\Big) \\
&= 2\sum_{v_iv_j \in E(G)} \Big(4D_G(v_iv_j) + D'_G(v_iv_j) + 2\big(d_G(v_i) + d_G(v_j)\big) + n\Big) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n} \Big(4D'_G(v_i) + D_G(v_i) + n - 1 + 4m\Big) \\
&= [16W_e(G) + 2W_{ve}(G) + 4M_1(G) + 2nm] + [2W_{ve}(G) + W(G) + \binom{n}{2} + 2nm] \\
&= 16W_e(G) + 4W_{ve}(G) + W(G) + 4M_1(G) + \binom{n}{2} + 4nm,
\end{aligned}
$$

from which (2) follows. □

THEOREM 3.2. *The edge-hyper-Wiener index of $\mathcal{SD}[G]$ is given by*

$$\begin{aligned} WW_e(\mathcal{SD}[G]) =& 16WW_e(G) + 2WW_{ve}(G) + WW(G) + 4W_{ve}(G) \\ &+ W(G) + 8M_1(G) + \binom{n}{2} + 4nm - 6m. \end{aligned} \tag{5}$$

*Proof.* Let $v_iv_j \in E(G)$. Then

$$\begin{aligned} D^{(2)}_{\mathcal{SD}[G]}(x_ix_j) = & \sum_{x_rx_s\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_ix_j, x_rx_s)^2 + \Big[d_{\mathcal{SD}[G]}(x_ix_j, y_iy_j)^2 \\ &+ \sum_{\substack{y_iy_s\in E(\mathcal{SD}[G]):\\ s\neq j}} d_{\mathcal{SD}[G]}(x_ix_j, y_iy_s)^2 + \sum_{\substack{y_ry_j\in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_ix_j, y_ry_j)^2 \\ &+ \sum_{\substack{y_ry_s\in E(\mathcal{SD}[G]):\\ r,s\neq i,j}} d_{\mathcal{SD}[G]}(x_ix_j, y_ry_s)^2\Big] + \Big[\sum_{\substack{x_ry_i\in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_i)^2 \\ &+ \sum_{\substack{x_ry_j\in E(\mathcal{SD}[G]):\\ r\neq j}} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_j)^2 + \sum_{\substack{x_ry_s\in E(\mathcal{SD}[G]):\\ s\neq r,i,j}} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_s)^2\Big] \\ &+ \sum_{r=1}^{n} d_{\mathcal{SD}[G]}(x_ix_j, x_ry_r)^2. \end{aligned}$$

Using Lemma 2.5 and by the same reasoning as in the proof of Theorem 3.1, we get

$$D^{(2)}_{\mathcal{SD}[G]}(x_ix_j) = 4D^{(2)}_G(v_iv_j) + D'^{(2)}_G(v_iv_j) + 2D'_G(v_iv_j) + 6\big(d_G(v_i)+d_G(v_j)\big) + n - 6.$$

Similarly, for $1 \le i \le n$, we have

$$\begin{aligned} D^{(2)}_{\mathcal{SD}[G]}(x_iy_i) = & \sum_{x_rx_s\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_iy_i, x_rx_s)^2 + \sum_{y_ry_s\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_iy_i, y_ry_s)^2 \\ &+ \sum_{\substack{x_ry_s\in E(\mathcal{SD}[G]):\\ r\neq s}} d_{\mathcal{SD}[G]}(x_iy_i, x_ry_s)^2 + \sum_{\substack{x_ry_r\in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_iy_i, x_ry_r)^2. \end{aligned}$$

By the same way as in the proof of Theorem 3.1, we get

$$D^{(2)}_{\mathcal{SD}[G]}(x_iy_i) = 4D'^{(2)}_G(v_i) + D^{(2)}_G(v_i) + 8D'_G(v_i) + 2D_G(v_i) + n - 1 + 4m.$$

Hence

$$\begin{aligned} W^{(2)}_e(\mathcal{SD}[G]) =& \frac{1}{2}\Big(4\sum_{x_ix_j\in E(\mathcal{SD}[G])} D^{(2)}_{\mathcal{SD}[G]}(x_ix_j) + \sum_{i=1}^{n} D^{(2)}_{\mathcal{SD}[G]}(x_iy_i)\Big) \\ =& 2\sum_{v_iv_j\in E(G)} \Big(4D^{(2)}_G(v_iv_j) + D'^{(2)}_G(v_iv_j) + 2D'_G(v_iv_j) + 6\big(d_G(v_i)+d_G(v_j)\big) + n - 6\Big) \\ &+ \frac{1}{2}\sum_{i=1}^{n} \Big(4D'^{(2)}_G(v_i) + D^{(2)}_G(v_i) + 8D'_G(v_i) + 2D_G(v_i) + n - 1 + 4m\Big) \\ =& 16W^{(2)}_e(G) + 2W^{(2)}_{ve}(G) + 4W_{ve}(G) + 12M_1(G) + 2m(n-6) \end{aligned}$$

$$+2W_{ve}^{(2)}(G)+W^{(2)}(G)+4W_{ve}(G)+2W(G)+\binom{n}{2}+2nm$$
$$=16W_e^{(2)}(G)+4W_{ve}^{(2)}(G)+8W_{ve}(G)+W^{(2)}(G)+2W(G)$$
$$+12M_1(G)+\binom{n}{2}+4nm-12m.$$

Now by definition of the edge-hyper-Wiener index and (2), we get

$$WW_e(\mathcal{SD}[G]) = \frac{1}{2}\Big(W_e(\mathcal{SD}[G])+W_e^{(2)}(\mathcal{SD}[G])\Big)$$
$$=\frac{1}{2}\Big(16W_e(G)+4W_{ve}(G)+W(G)+4M_1(G)+\binom{n}{2}+4nm+16W_e^{(2)}(G)$$
$$+4W_{ve}^{(2)}(G)+8W_{ve}(G)+W^{(2)}(G)+2W(G)+12M_1(G)+\binom{n}{2}+4nm-12m\Big)$$
$$=16WW_e(G)+2WW_{ve}(G)+WW(G)+4W_{ve}(G)+W(G)+8M_1(G)+\binom{n}{2}+4nm-6m,$$

from which (5) follows. □

THEOREM 3.3. *The edge-degree distance of $\mathcal{SD}[G]$ is given by*

$$DD_e(\mathcal{SD}[G]) =32DD_e(G)+8DD_{ev}(G)+16DD_{ve}(G)+4DD(G)+16W_{ve}(G) +128W_e(G)+16F(G)+32M_2(G)+8nM_1(G)+8m(n-1+4m). \quad (6)$$

*Proof.* By definition of the edge-degree distance and Definition 2.1, we get

$$DD_e(\mathcal{SD}[G]) =4\sum_{x_ix_j\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_ix_j)D_{\mathcal{SD}[G]}(x_ix_j) + \sum_{i=1}^{n} d_{\mathcal{SD}[G]}(x_iy_i)D_{\mathcal{SD}[G]}(x_iy_i).$$

Now by Lemmas 2.4 and 2.5 and (3) and (4), we obtain

$$DD_e(\mathcal{SD}[G]) = 4\sum_{v_iv_j\in E(G)} \big(2d_G(v_iv_j)+4\big)\Big(4D_G(v_iv_j)+D'_G(v_iv_j)+2\big(d_G(v_i)+d_G(v_j)\big)+n\Big)$$
$$+\sum_{i=1}^{n} 4d_G(v_i)\Big(4D'_G(v_i)+D_G(v_i)+n-1+4m\Big)$$
$$=4\sum_{v_iv_j\in E(G)}\Big(8d_G(v_iv_j)D_G(v_iv_j)+2d_G(v_iv_j)D'_G(v_iv_j)+4d_G(v_iv_j)\big(d_G(v_i)+d_G(v_j)\big)$$
$$+2nd_G(v_iv_j)+16D_G(v_iv_j)+4D'_G(v_iv_j)+8(d_G(v_i)+d_G(v_j))+4n\Big)$$
$$+\sum_{i=1}^{n}\Big(16d_G(v_i)D'_G(v_i)+4d_G(v_i)D_G(v_i)+4(n-1+4m)d_G(v_i)\Big).$$

Using the fact that $d_G(v_iv_j) = d_G(v_i)+d_G(v_j) - 2$, we obtain

$$DD_e(\mathcal{SD}[G]) = 4\Big(8DD_e(G)+2DD_{ev}(G)+4(F(G)+2M_2(G) - 2M_1(G))$$
$$+2n(M_1(G) - 2m)+32W_e(G)+4W_{ve}(G)+8M_1(G)+4nm\Big)$$

$$+16DD_{ve}(G)+4DD(G)+8m(n-1+4m)$$
$$=32DD_e(G)+8DD_{ev}(G)+16DD_{ve}(G)+4DD(G)+16W_{ve}(G)$$
$$+128W_e(G)+16F(G)+32M_2(G)+8nM_1(G)+8m(n-1+4m),$$

from which (6) follows. □

THEOREM 3.4. *The edge-Gutman index of $\mathcal{SD}[G]$ is given by*

$$\begin{aligned} Gut_e(\mathcal{SD}[G]) =& 64Gut_e(G) + 32Gut_{ve}(G) + 16Gut(G) + 128DD_e(G) \\ &+ 64DD_{ve}(G) + 256W_e(G) + 96F(G) + 32M_2(L(G)) \\ &+ 192M_2(G) + (64m-200)M_1(G) + 32m(m+4). \end{aligned} \tag{7}$$

*Proof.* Let $v_iv_j \in E(G)$. Then

$$S_{\mathcal{SD}[G]}(x_ix_j) = \sum_{x_rx_s\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_rx_s)d_{\mathcal{SD}[G]}(x_ix_j, x_rx_s)$$
$$+ \Big[d_{\mathcal{SD}[G]}(y_iy_j)d_{\mathcal{SD}[G]}(x_ix_j, y_iy_j) + \sum_{\substack{y_iy_s\in E(\mathcal{SD}[G]):\\ s\neq j}} d_{\mathcal{SD}[G]}(y_iy_s)d_{\mathcal{SD}[G]}(x_ix_j, y_iy_s)$$
$$+ \sum_{\substack{y_ry_j\in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(y_ry_j)d_{\mathcal{SD}[G]}(x_ix_j, y_ry_j) + \sum_{\substack{y_ry_s\in E(\mathcal{SD}[G]):\\ r,s\neq i,j}} d_{\mathcal{SD}[G]}(y_ry_s)d_{\mathcal{SD}[G]}(x_ix_j, y_ry_s)\Big]$$
$$+ \Big[\sum_{\substack{x_ry_i\in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_ry_i)d_{\mathcal{SD}[G]}(x_ix_j, x_ry_i) + \sum_{\substack{x_ry_j\in E(\mathcal{SD}[G]):\\ r\neq j}} d_{\mathcal{SD}[G]}(x_ry_j)d_{\mathcal{SD}[G]}(x_ix_j, x_ry_j)$$
$$+ \sum_{\substack{x_ry_s\in E(\mathcal{SD}[G]):\\ s\neq r,i,j}} d_{\mathcal{SD}[G]}(x_ry_s)d_{\mathcal{SD}[G]}(x_ix_j, x_ry_s)\Big] + \sum_{r=1}^{n} d_{\mathcal{SD}[G]}(x_ry_r)d_{\mathcal{SD}[G]}(x_ix_j, x_ry_r).$$

Now by Lemmas 2.4 and 2.5, we get

$$S_{\mathcal{SD}[G]}(x_ix_j) = \sum_{v_rv_s\in E(G)} \big(2d_G(v_rv_s)+4\big)d_G(v_iv_j, v_rv_s)$$
$$+\Big[\big(2d_G(v_iv_j)+4\big)\big(d_G(v_iv_j, v_iv_j)+2\big)+ \sum_{\substack{v_iv_s\in E(G):\\ s\neq j}} \big(2d_G(v_iv_s)+4\big)\big(d_G(v_iv_j, v_iv_s)+1\big)$$
$$+ \sum_{\substack{v_rv_j\in E(G):\\ r\neq i}} \big(2d_G(v_rv_j)+4\big)\big(d_G(v_iv_j, v_rv_j)+1\big)+ \sum_{\substack{v_rv_s\in E(G):\\ r,s\neq i,j}} \big(2d_G(v_rv_s)+4\big)d_G(v_iv_j, v_rv_s)\Big]$$
$$+\Big[\sum_{v_rv_i\in E(G)} \big(2d_G(v_rv_i)+4\big)\big(d_G(v_iv_j, v_rv_i)+1\big)+ \sum_{v_rv_j\in E(G)} \big(2d_G(v_rv_j)+4\big)\big(d_G(v_iv_j, v_rv_j)+1\big)$$
$$+ \sum_{\substack{v_rv_s\in E(G):\\ s\neq i,j}} \big(2d_G(v_rv_s)+4\big)d_G(v_iv_j, v_rv_s)\Big]+ \sum_{r=1}^{n} 4d_G(v_r)\big(d_G(v_iv_j, v_r)+1\big)$$
$$=\big(2S_G(v_iv_j)+4D_G(v_iv_j)\big)+\big[4(d_G(v_iv_j)+2)+2S_G(v_iv_j)+4D_G(v_iv_j)$$

$$+2\delta_G(v_iv_j)+4(d_G(v_i)-1)+4(d_G(v_j)-1)\big]+\big[4S_G(v_iv_j)+8D_G(v_iv_j)$$
$$+2\delta_G(v_iv_j)+4d_G(v_iv_j)+4(d_G(v_i)+d_G(v_j))\big]+4S'_G(v_iv_j)+8m$$
$$=8S_G(v_iv_j)+4S'_G(v_iv_j)+16D_G(v_iv_j)+16\big(d_G(v_i)+d_G(v_j)\big)+4\delta_G(v_iv_j)+8m-16.$$

By symmetry, for each edge $v_iv_j \in E(G)$, we have

$$S_{\mathcal{SD}[G]}(y_iy_j) = S_{\mathcal{SD}[G]}(x_iy_j) = S_{\mathcal{SD}[G]}(x_jy_i) = S_{\mathcal{SD}[G]}(x_ix_j)$$
$$=8S_G(v_iv_j)+4S'_G(v_iv_j)+16D_G(v_iv_j)+16\big(d_G(v_i)+d_G(v_j)\big)+4\delta_G(v_iv_j)+8m-16.$$

Similarly for $1 \leq i \leq n$,

$$\begin{aligned} S_{\mathcal{SD}[G]}(x_iy_i) = &\sum_{x_rx_s\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_rx_s)d_{\mathcal{SD}[G]}(x_iy_i, x_rx_s) \\ &+ \sum_{y_ry_s\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(y_ry_s)d_{\mathcal{SD}[G]}(x_iy_i, y_ry_s) \\ &+ \sum_{\substack{x_ry_s\in E(\mathcal{SD}[G]):\\ r\neq s}} d_{\mathcal{SD}[G]}(x_ry_s)d_{\mathcal{SD}[G]}(x_iy_i, x_ry_s) \\ &+ \sum_{\substack{x_ry_r\in E(\mathcal{SD}[G]):\\ r\neq i}} d_{\mathcal{SD}[G]}(x_ry_r)d_{\mathcal{SD}[G]}(x_iy_i, x_ry_r). \end{aligned}$$

Now from Lemmas 2.4 and 2.5, we get

$$S_{\mathcal{SD}[G]}(x_iy_i) = \sum_{v_rv_s\in E(G)} \big(2d_G(v_rv_s)+4)\big)\big(d_G(v_rv_s, v_i)+1\big)$$
$$+ \sum_{v_rv_s\in E(G)} \big(2d_G(v_rv_s)+4)\big)\big(d_G(v_rv_s, v_i)+1\big)$$
$$+ 2\sum_{v_rv_s\in E(G)} \big(2d_G(v_rv_s)+4)\big)\big(d_G(v_rv_s, v_i)+1\big)$$
$$+ \sum_{\substack{r=1,\\ r\neq i}}^{n} 4d_G(v_r)\big(d_G(v_i, v_r)+1\big)$$
$$=4\big(2S'_G(v_i)+2(M_1(G)-2m)+4D'_G(v_i)+4m\big)+4\big(S_G(v_i)+2m-d_G(v_i)\big)$$
$$=8S'_G(v_i)+4S_G(v_i)+16D'_G(v_i)-4d_G(v_i)+8M_1(G)+8m.$$

By definition of the edge-Gutman index and Lemmas 2.4 and 2.5,

$$Gut_e(\mathcal{SD}[G])$$
$$=\frac{1}{2}\Big(4\sum_{x_ix_j\in E(\mathcal{SD}[G])} d_{\mathcal{SD}[G]}(x_ix_j)S_{\mathcal{SD}[G]}(x_ix_j)+\sum_{i=1}^{n} d_{\mathcal{SD}[G]}(x_iy_i)S_{\mathcal{SD}[G]}(x_iy_i)\Big)$$
$$=2\sum_{v_iv_j\in E(G)} \big(2d_G(v_iv_j)+4\big)\Big(8S_G(v_iv_j)+4S'_G(v_iv_j)+16D_G(v_iv_j)$$
$$+16\big(d_G(v_i)+d_G(v_j)\big)+4\delta_G(v_iv_j)+8m-16\Big)$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=1}^{n} 4d_G(v_i)\Big(8S'_G(v_i)+4S_G(v_i)+16D'_G(v_i)-4d_G(v_i)+8M_1(G)+8m\Big)\\
=&2\sum_{v_iv_j\in E(G)}\Big(16d_G(v_iv_j)S_G(v_iv_j)+8d_G(v_iv_j)S'_G(v_iv_j)+32d_G(v_iv_j)D_G(v_iv_j)\\
&+32d_G(v_iv_j)\big(d_G(v_i)+d_G(v_j)\big)+8d_G(v_iv_j)\delta_G(v_iv_j)+2(8m-16)d_G(v_iv_j)+32S_G(v_iv_j)\\
&+16S'_G(v_iv_j)+64D_G(v_iv_j)+64\big(d_G(v_i)+d_G(v_j)\big)+16\delta_G(v_iv_j)+4(8m-16)\Big)\\
&+2\sum_{i=1}^{n}\Big(8d_G(v_i)S'_G(v_i)+4d_G(v_i)S_G(v_i)+16d_G(v_i)D'_G(v_i)-4d_G(v_i)^2\\
&+8d_G(v_i)M_1(G)+8md_G(v_i)\Big).
\end{aligned}
$$

Using the identity $d_G(v_iv_j) = d_G(v_i) + d_G(v_j) - 2$ and (1), we obtain

$$
\begin{aligned}
Gut_e(\mathcal{SD}[G]) =&64Gut_e(G) + 16Gut_{ve}(G) + 64DD_e(G) + 64(F(G) + 2M_2(G) - 2M_1(G))\\
&+ 32M_2(L(G)) + 4(8m - 16)(M_1(G) - 2m) + 64DD_e(G) + 32DD_{ve}(G)\\
&+ 256W_e(G) + 128M_1(G)) + 32(F(G) + 2M_2(G) - 4M_1(G) + 4m)\\
&+ 8m(8m - 16) + 16Gut_{ve}(G) + 16Gut(G) + 32DD_{ve}(G) - 8M_1(G)\\
&+ 32mM_1(G) + 32m^2\\
=&64Gut_e(G) + 32Gut_{ve}(G) + 16Gut(G) + 128DD_e(G) + 64DD_{ve}(G)\\
&+ 256W_e(G) + 96F(G) + 32M_2(L(G)) + 192M_2(G)\\
&+ (64m - 200)M_1(G) + 32m(m + 4),
\end{aligned}
$$

from which (7) follows. □

## 4. Concluding remarks

In this paper we have presented some formulas for the distance between edges in strong double graphs. Our findings have been applied in computing exact values for a number of edge-distance- related topological indices namely the edge-Wiener index, the edge-hyper-Wiener index, the edge- degree distance and the edge-Gutman index of this family of graph transformations. The obtained results show the connection between the above invariants of the strong double graph and the corresponding invariants of its parent graph. Vertex-degree-based topological indices of the strong double graphs have been well studied in the literature. However, topological indices from other categories, such as distance-based, transmission-based, eccentricity-based, spectrum-based, and eigenvalue-based, have not been sufficiently investigated and it is useful to consider them in future research. In addition, various other families of graph transformations such as line graphs, subdivision graphs, total graphs, semi-total point graphs, semi-total line graphs, double graphs, and Mycielski graphs are proposed to be studied under different categories of topological indices.

Acknowledgement. The author wish to thank the respected referee for his/her valuable comments which improved the presentation of the paper.

References


**[1]** N. Akhter, M. Amin, M. K. Jamil, W. Gao, *Some distance-based topological indices of strong double graphs*, Asian J. Math. Appl., **2018** (2018), ama0475.

**[2]** M. A. Ali, M. S. Sardar, I. Siddique, D. Alrowaili, *Vertex-based topological indices of double and strong double graph of Dutch windmill graph*, J. Chem., **2021** (2021), 7057412.

**[3]** Y. Alizadeh, A. Iranmanesh, T. Došlić, M. Azari, *The edge Wiener index of suspensions, bottlenecks, and thorny graphs*, Glas. Mat. Ser. III, **49(69)** (2014), 1–12.

**[4]** M. Azari, H. Divanpour, *Splices, links, and their edge-degree distances*, Trans. Comb., **6(4)** (2017), 29–42.

**[5]** M. Azari, A. Iranmanesh, *Clusters and various versions of Wiener-type invariants*, Kragujevac J. Math., **39(2)** (2015), 155–171.

**[6]** M. Azari, A. Iranmanesh, A. Tehranian, *Two topological indices of three chemical structures*, MATCH Commun. Math. Comput. Chem., **69(1)** (2013), 69–86.

**[7]** P. Dankelman, I. Gutman, S. Mukwembi, H. C. Swart, *The edge Wiener index of a graph*, Discrete Math., **309(10)** (2009), 3452–3457.

**[8]** A. Dobrynin, A. A. Kochetova, *Degree distance of a graph: A degree analogue of the Wiener index*, J. Chem. Inf. Comput. Sci., **34(5)** (1994), 1082–1086.

**[9]** B. Furtula, I. Gutman, *A forgotten topological index*, J. Math. Chem., **53(4)** (2015), 1184–1190.

**[10]** I. Gutman, *Selected properties of the Schultz molecular topological index*, J. Chem. Inf. Comput. Sci., **34(5)** (1994), 1087–1089.

**[11]** I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals. Total $\pi$-electron energy of alternant hydrocarbons*, Chem. Phys. Lett., **17(4)** (1972), 535–538.

**[12]** I. Gutman, B. Ruščić, N. Trinajstić, C. F. Wilcox, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, J. Chem. Phys., **62(9)** (1975), 3399–3405.

**[13]** M. Imran, S. Akhter, *Degree-based topological indices of double graphs and strong double graphs*, Discrete Math. Alg. Appl., **9(5)** (2017), 1750066.

**[14]** A. Iranmanesh, M. Azari, *Edge-Wiener descriptors in chemical graph theory: A survey*, Curr. Org. Chem., **19(3)** (2015), 219–239.

**[15]** A. Iranmanesh, I. Gutman, O. Khormali, A. Mahmiani, *The edge versions of Wiener index*, MATCH Commun. Math. Computt. Chem., **61(3)** (2009), 663–672.

**[16]** A. Iranmanesh, O. Khormali, A. Ahmadi, *Generalized edge-Schultz indices of some graphs*, MATCH Commun. Math. Comput. Chem., **65(1)** (2011), 93–112.

**[17]** A. Iranmanesh, A. Soltani Kafrani, O. Khormali, *A new version of hyper-Wiener index*, MATCH Commun. Math. Comput. Chem., **65(1)** (2011), 113–122.

**[18]** M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, S. G. Wagner, *Some new results on distance-based graph invariants*, European J. Combin., **30(5)** (2009), 1149–1163.

**[19]** D. J. Klein, I. Lukovits, I. Gutman, *On the definition of the hyper-Wiener index for cycle-containing structures*, J. Chem. Inf. Comput. Sci., **35(1)** (1995), 50–52.

**[20]** E. Munarini, C. Perelli Cippo, A. Scagliola, N. Zagaglia Salvi, *Double graphs*, Discrete Math., **308(2-3)** (2008), 242–254.

**[21]** P. Ž. Pleteršek, *The edge-Wiener index and the edge-hyper-Wiener index of phenylenes*, Discrete Appl. Math., **255(2-3)** (2019), 326–333.

**[22]** M. Rafiullah, H. M. A. Siddiqui, M. K. Siddiqui, M. Dhlamini, *On degree-based topological indices for strong double graphs*, J. Chem., **2021** (2021), 4852459.

**[23]** H. S. Ramane, I. Gutman, K. Bhajantri, D. V. Kitturmath, *Sombor index of some graph transformations*, Commun. Comb. Optim., **8(1)** (2023), 193–205.

**[24]** M. Randić, *Novel molecular descriptor for structure-property studies*, Chem. Phys. Lett., **211(10)** (1993), 478–483.

**[25]** M. S. Sardar, M. A. Ali, F. Ashraf, M. Cancan, *On topological indices of double and strong double graph of silicon carbide* $Si_2C_3 - I[p, q]$, Eurasian Chem. Commun., **5(1)** (2023), 37–49.

**[26]** M. S. Sardar, I. Siddique, D. Alrowaili, M. A. Ali, S. Akhtar, *Computation of topological indices of double and strong double graphs of circumcoronene series of benzenoid*, J. Math., **2022** (2022), 5956802.

**[27]** N. Tratnik, *A method for computing the edge-hyper-Wiener index of partial cubes and an algorithm for benzenoid systems*, Appl. Anal. Discrete Math., **12(1)** (2018), 126–142.

**[28]** G. Wang, Y. Liu, J. Wei, J.-B. Liu, *The edge-hyper-Wiener index of zigzag single-walled nanotubes*, Polycycl. Aromat. Compd., **43(2)** (2023), 1509–1523.

**[29]** H. Wiener, *Structural determination of paraffin boiling points*, J. Am. Chem. Soc., **69(1)** (1947), 17–20.

**[30]** H. Yousefi-Azari, M. H. Khalifeh, A. R. Ashrafi, *Calculating the edge Wiener and edge Szeged indices of graphs*, J. Comput. Appl. Math., **235(16)** (2011), 4866–4870.

(received 19.04.2022; in revised form 30.12.2023; available online 21.08.2024)

Department of Mathematics, Kazerun Branch, Islamic Azad University, P. O. Box: 73135-168, Kazerun, Iran

*E-mail*: mahdie.azari@gmail.com, mahdieh.azari@iau.ac.ir

ORCID iD: https://orcid.org/0000-0002-0919-0598