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EXISTENCE AND NON EXISTENCE OF SOLUTIONS FOR A BI-NONLOCAL PROBLEM

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Abstract. The aim of this paper is to investigate the existence and non-existence of non-trivial weak solutions to a bi-nonlocal problem under sufficient conditions by using the variational arguments.

1. Introduction and main result

In recent years, the study of differential equations and variational problems with nonlocal operators has emerged in many fields such as finance, optimization, continuum mechanics, phase transition phenomena, population dynamics and game theory [4, 13, 18, 19].

Here we are interested in the following bi-nonlocal problem,

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u=\lambda f(x,u)\left[\int_\Omega F(x,u)dx\right]^r+\mu g(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded smooth domain, $f, g : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions satisfying conditions given later. $F(x,u) = \int_0^u f(x,s)ds$, λ, μ and r are real parameters with $r > 0$. $p \in C(\overline{\Omega})$ with $N > p(x) > 1$ and

$$\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u) = \sum_{i=1}^{N}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right)$$

is the $p(x)$-Laplacian operator.

This type of problem arises in the modeling of biological systems where u describes a process that depends on the average of itself, such as the population density [2, 5]. Furthermore, bi-nonlocal problems are related to the stationary version of a model, the so-called Kirchhoff equation, which was introduced by Kirchhoff in 1883 [16]. The various Kirchhoff-type equations have been studied (see e.g. [3, 9, 14, 15]).

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The problem (1) was investigated by Corrêa et al. [7]. For the special case that $\mu = 0$, the authors proved the existence of infinitely many solutions for λ strictly positive, for the following bi-nonlocal problem

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = \lambda f(x,u)\left[\int_\Omega F(x,u)dx\right]^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

After Corrêa et al. [8], when $g(x,u) = |u|^{q(x)-2}u$ and $\mu = 1$, under appropriate conditions and by a version of the Concentration Compactness Principle, investigated the existence of a continuous family of eigenvalues considering different classes of the Kirchhoff term M. More precisely, they showed the existence of a positive $\overline{\lambda}$, so that the problem (1) has a non-trivial solution for all $\lambda > \overline{\lambda}$.

Motivated by the work mentioned above, we examine the problem (1) under more general conditions than in [8]. We discuss the existence and non-existence of weak solutions for the problem (1), where we obtain infinitely many solutions by applying the Fountain theorem and the Krasnoselskii genus.

In this paper we assume the following hypotheses M, f and g : there are positive constants $m_0, m_1, A_1,\ A_2,\ B_1,\ B_2$ and functions $\alpha(x), \beta(x) \in C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}); h(x) > 1, \forall x \in \overline{\Omega}\}$, such that

(M1) $m_0 \le M(t) \le m_1$.

(f$_1$) $A_1 t^{\beta(x)-1} \le f(x,t) \le A_2 t^{\beta(x)-1}$. (f$_2$) $f(x,t) = -f(x,-t)$ for all $(x,t) \in (\Omega, \mathbb{R})$.

(g$_1$) $B_1 t^{\alpha(x)-1} \le g(x,t) \le B_2 t^{\alpha(x)-1}$. (g$_2$) $g(x,t) = -g(x,-t)$ for all $(x,t) \in (\Omega, \mathbb{R})$.

Problems of the form (1), are associated with the energy functional

$$J_{\lambda,\mu}(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} - \mu\int_\Omega G(x,u)dx,$$

for all $u \in W_0^{1,p(x)}(\Omega)$, where $W_0^{1,p(x)}(\Omega)$ is the generalized Lebesgue-Sobolev space whose precise definition and properties are established in Section 2 and $\widehat{M}(t) = \int_0^t M(s)ds$.

The functional $J_{\lambda,\mu}$ is differentiable and its Fréchet derivative is given by

$$\begin{aligned} J'_{\lambda,\mu}(u)(v) = {} & M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx \\ & - \lambda\left[\int_\Omega F(x,u)dx\right]^r \int_\Omega f(x,u)uv\, dx - \mu\int_\Omega g(x,u)uv\, dx \end{aligned}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Thus, the weak solution of the problem (1), coincides with the critical point of $J_{\lambda,\mu}$.

Let us define, for every $x \in \Omega$,

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geqslant N. \end{cases}$$

Now we can present our main result.

THEOREM 1.1. *Suppose that* $\alpha^- < \alpha(x) < \dfrac{B_2}{B_1}\alpha^+ < p^- < p(x) < p^+ < \beta^- < \beta(x)p^+$.
(i) For every $\lambda > 0$, $\mu \in \mathbb{R}$, *with (M1), (f_1), (f_2), (g_1) and (g_2) satisfied, assuming*

$$\frac{m_1 p^+}{m_0} < \left(\frac{A_1}{A_2}\right)^{r+1} \frac{(\beta^-)^{r+1}(r+1)}{(\beta^+)^r}.$$

Then problem (1) *has a sequence of weak solutions* (u_k) *such that* $J_{\lambda,\mu}(u_k) \to +\infty$ *as* $k \to +\infty$.

(ii) For every $\lambda > 0$, $\mu > 0$, *where (M1), (f_1), (f_2), (g_1) and (g_2) are satisfied, assume*

$$\frac{m_1 p^+}{m_0} < \left(\frac{A_1}{A_2}\right)^{r+1} \frac{(\beta^-)^{r+1}(r+1)}{(\beta^+)^r}.$$

Then the problem (1) *has a sequence of weak solutions* (u_k) *such that* $J_{\lambda,\mu}(u_k) < 0$ *and* $J_{\lambda,\mu}(u_k) \to 0$ *as* $k \to +\infty$.

(iii) For every $\lambda < 0$, $\mu > 0$, *with (M1), (f_1), (f_2), (g_1) and (g_2) satisfied, the problem* (1) *has infinitely many solutions.*

(iv) For every $\lambda < 0$, $\mu < 0$, *with (M1), (f_1) and (g_1) satisfied,* (1) *has no non-trivial weak solution.*

The rest of this paper is structured as follows. The Section 2 contains some properties concerning the generalized Lebesgue and Sobolev spaces and embedding results. The proofs of our results are presented in Section 3.

2. Preliminaries on variable exponent spaces

To study $p(x)$-Laplacian problems, we need some theories about the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For details see [10, 12].

Define $\forall h \in C_+(\overline{\Omega})$, $h^- = \min_{x\in\overline{\Omega}} h(x) \leq h^+ = \max_{x\in\overline{\Omega}} h(x)$. For $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebegue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \quad \text{measurable}; \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\left\{ \eta > 0 : \int_\Omega \left|\frac{u}{\eta}\right|^{p(x)} dx \leq 1 \right\},$$

which is a separable and reflexive Banach space.

PROPOSITION 2.1 ([12]). *(i) The space* $(L^{p(x)}(\Omega), |u|_{p(x)})$ *is a separable, uniformly convex Banach space and its dual space is* $L^{q(x)}$, *where* $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$\left| \int_\Omega uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}.$$

(ii) If $p_1(x), p_2(x) \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$ $\forall x \in \overline{\Omega}$, *then* $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ *and the embedding is continuous.*

PROPOSITION 2.2 ([12]). *Let* $\rho(u) = \int_\Omega |u|^{p(x)}dx$. *For* $u, u_n \in L^{p(x)}(\Omega)$, *we have*

1. $|u|_{p(x)} < 1$ *(resp* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp* $= 1, > 1$*).*

2. $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \leqslant \rho(u) \leqslant \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.

3. $|u_n(x)|_{p(x)} \to 0$ *(resp.* $\to \infty$*)* $\Leftrightarrow \rho(u_n) \to 0$*(resp.* $\to \infty$*).*

The Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \left\{ u : \Omega \subset \mathbb{R}^N \to \mathbb{R} : u \in L^{p(x)}(\Omega), \nabla u \in (L^{p(x)}(\Omega))^N \right\},$$

and is equipped with the norm $\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$. Then $W^{1,p(x)}(\Omega)$ also becomes a separable, reflexive and Banach space. We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

PROPOSITION 2.3 ([12, Sobolev embedding]). *Suppose that* Ω *is a bounded smooth domain in* $\mathbb{R}^N$ *and* $p, q \in C_+(\overline{\Omega})$ *such that* $1 \le q(x) \le p^*(x)$ *for all* $x \in \overline{\Omega}$, *then there is a continuous embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$. *If we replace* $\le$ *by* $<$, *the embedding is compact.*

PROPOSITION 2.4 ([12, Poincaré inequality]). *If* $u \in W^{1,p(x)}(\Omega)$, *then* $|u|_{p(x)} \le C|\nabla u|_{p(x)}$, *where* C *is a constant that does not depend on* u.

REMARK 2.5. By Proposition 2.4 we know that $|\nabla u|_{p(x)}$ and $|u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. It follows that we work on $W_0^{1,p(x)}(\Omega)$ with the norm $\|u\| = |\nabla u|_{p(x)}$.

PROPOSITION 2.6 ([11]). *Denote*

$$\langle L_{p(x)}(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx \quad \forall u, v \in W_0^{1,p(x)}(\Omega).$$

(i) $L_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ *is a continuous, bounded, and strictly monotone operator.*

(ii) $L_{p(x)}$ *satisfies condition* (S_+), *namely,* $u_n \rightharpoonup u$ *and* $\limsup\langle L_{p(x)}(u_n), u_n - u\rangle \le 0$, *imply* $u_n \to u$ *in* $W_0^{1,p(x)}(\Omega)$.

(iii) $L_{p(x)} : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ *is a homeomorphism.*

To prove our main results, we need to apply the following Fountain theorem and the Krasnoselskii's genus.

Let X be a reflexive and separable Banach space, then there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that $X = \overline{\mathrm{span}\{e_j : j = 1, 2, 3, 4, \ldots\}}$, $X^* = \overline{\mathrm{span}\{e_j^* : j = 1, 2, 3, 4, \ldots\}}$, with $\langle e_i, e_j^*\rangle = \delta_{ij}$, where δ_{ij} denotes Kronecker's delta symbol. Define

$$X_j = \mathrm{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{3}$$

Then we have the following proposition.

PROPOSITION 2.7. *If* $\beta(x), \alpha(x) \in C_+(\overline{\Omega}), \beta(x), \alpha(x) < p^*(x)$ *for* $x \in \overline{\Omega}$*, let*

$$\beta_k = \sup\{|u|_{\beta(x)} : \|u\| = 1, u \in Z_k\}, \quad \theta_k = \sup\{|u|_{\alpha(x)} : \|u\| = 1, u \in Z_k\},$$

then $\lim_{k\to\infty} \beta_k = 0$, $\lim_{k\to\infty} \theta_k = 0$.

LEMMA 2.8 ([20, Fountain theorem]). *Let*
(H1) $I \in C^1(X, \mathbb{R})$ *be an even functional, where* $(X; \|.\|)$ *is a separable and reflexive Banach space, the subspaces* X_k, Y_k *and* Z_k *are defined by* (3).

If for each $k \in \mathbb{N}$*, there exist* $\rho_k > r_k > 0$ *such that*

(H2) $\inf\{I(u) : u \in Z_k; \|u\| = r_k\} \to +\infty$ *as* $k \to +\infty$.

(H3) $\max\{I(u) : u \in Y_k; \|u\| = \rho_k\} \leq 0$.

(H4) I satisfies the (PS)*-condition for every* $c > 0$.
Then I has an unbounded sequence of critical points.

LEMMA 2.9 ([20, Dual Fountain theorem]). *Assuming (H1) is satisfied and there is* $k_0 > 0$ *such that for every* $k \geq k_0$ *there exist* $\rho_k > r_k > 0$ *such that*
(L1) $a_k = \inf\{I(u) : u \in Z_k; \|u\| = \rho_k\} \geq 0$.

(L2) $b_k = \max\{I(u) : u \in Y_k; \|u\| = r_k\} < 0$.

(L3) $d_k = \inf\{I(u) : u \in Z_k; \|u\| \leq \rho_k\} \to 0$ *as* $k \to +\infty$

(L4) I satisfies the condition $(PS)_c^*$ *for every* $c \in [d_{k_0}, 0)$.
Then I has a sequence of negative critical values converging to 0.

DEFINITION 2.10. We say that I satisfies the condition $(PS)_c^*$ (with respect to (Y_{n_j})), if every sequence $u_{n_j} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $I(u_{n_j}) \to c$ and $(I|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence that converges to a critical point of I.

In the following we recall some basic notions of Krasnoselskii's genus.

Let X be a real Banach space. Set $\Re = \{E \subset X \setminus \{0\} : E$ is compact and $E = -E\}$.

DEFINITION 2.11. Let $E \in \Re$ and $X = \mathbb{R}^k$. The genus $\gamma(E)$ of E is defined by

$$\gamma(E) = \min\left\{k \geq 1;\ \text{there exists an odd continuous mapping } \phi : E \to \mathbb{R}^k \setminus \{0\}\right\}.$$

If such a mapping does not exist for any $k > 0$, we set $\gamma(E) = \infty$. Note also that $\gamma(E) = 1$ if E is a subset consisting of finitely many pairs of points. Furthermore, $\gamma(\emptyset) = 0$. A typical example of a set of genus k is a set that is homeomorphic to a $(k-1)$ dimensional sphere via an odd mapping.

We now state some results about the Krasnoselskii's genus that are necessary for the present proof.

THEOREM 2.12. *Let* $X = \mathbb{R}^k$ *and* $\partial\Omega$ *be the boundary of an open, symmetric and bounded subset* $\Omega \subset \mathbb{R}^k$ *with* $0 \in \Omega$*. Then* $\gamma(\partial\Omega) = N$.

Corollary 2.13. $\gamma(S^{k-1}) = N - 1$.

Remark 2.14. If X is of infinite dimension and separable and S is the unit sphere in X, then $\gamma(S) = \infty$.

The following result obtained by Clarke [6] is the main idea we use in our proof.

Theorem 2.15. *Let $J \in C^1(X, \mathbb{R})$ be a functional satisfying the condition (PS). Let us further assume that the following hold.*
(i) J is bounded from below and even.

(ii) There exists a compact set $K \in \Re$ such that $\gamma(K) = k$ and $\sup_K J(u) < 0 = J(0)$. Then J has at least k pairs of distinct critical points, and their corresponding critical values are smaller than $J(0)$.

Further information on this topic can be found in the references [1, 17].
In this paper we use $c_i (i = 1, 2, \dots)$ to denote the general non-negative constants.

3. Proof of Theorem 1.1

We will use the Fountain theorem to prove (i), and the Dual Fountain theorem to prove (ii). We will use genus theory to prove (iii).

Proof. (i) First we verify that $J_{\lambda,\mu}$ satisfies the (PS) condition. Suppose that $(u_n) \subset W_0^{1,p(x)}(\Omega)$ is (PS) sequence, i.e.

$$J_{\lambda,\mu}(u_n) \to c_1 \quad \text{and} \quad J'_{\lambda,\mu}(u_n) \to 0. \tag{4}$$

Step 1. We prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. For convenience, assume that $\{u_n\}$ is unbounded in X. Thus to a passing a subsequence if necessary, we get $\|u_n\| > 1$ for n large enough. Take

$$\frac{m_1 p^+}{m_0} < \theta < \left(\frac{A_1}{A_2}\right)^{r+1} \frac{(\beta^-)^{r+1}(r+1)}{(\beta^+)^r}.$$

• For $\mu \leq 0$. From (4), (M1), (f_1) and (g_1), we have

$$\begin{aligned}
&c_1 + 1 + ||u_n|| \geq J_{\lambda,\mu}(u_n) - \frac{1}{\theta} J'_{\lambda,\mu}(u_n) u_n \\
&= \widehat{M}\left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x, u_n) dx\right]^{r+1} - \mu \int_\Omega G(x, u_n) dx \\
&- \frac{1}{\theta}\left(M\left(\int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx\right) \int_\Omega |\nabla u_n|^{p(x)} dx - \lambda \left[\int_\Omega F(x, u_n) dx\right]^r \int_\Omega f(x, u_n) u_n dx\right) \\
&+ \frac{\mu}{\theta} \int_\Omega g(x, u_n) u_n dx \\
&\geq (\frac{m_0}{p^+} - \frac{m_1}{\theta}) \int_\Omega |\nabla u_n|^{p(x)} dx + \lambda\left(\frac{A_1^{r+1}}{\theta(\beta^+)^r} - \frac{1}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\right)\left(\int_\Omega |u_n|^{\beta(x)} dx\right)^{r+1}
\end{aligned}$$

$$-\mu(\frac{B_1}{\alpha^+}-\frac{B_2}{\theta})\int_\Omega |u_n|^{\alpha(x)}dx \geq (\frac{m_0}{p^+}-\frac{m_1}{\theta})\|u_n\|^{p^-}$$

which is contradiction because $p^- > 1$. Hence $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

- For $\mu > 0$. From (4), (M1), (f_1) and (g_1), we have

$$\begin{aligned} c_1 + 1 + ||u_n|| &\geq J_{\lambda,\mu}(u_n) - \frac{1}{\theta}J'_{\lambda,\mu}(u_n)u_n \\ &\geq (\frac{m_0}{p^+}-\frac{m_1}{\theta})\int_\Omega |\nabla u_n|^{p(x)}dx - \mu(\frac{B_2}{\alpha^-}-\frac{B_1}{\theta})\int_\Omega |u_n|^{\alpha(x)}dx \\ &\quad + \lambda\left(\frac{A_1^{r+1}}{\theta(\beta^+)^r} - \frac{1}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\right)\left(\int_\Omega |u_n|^{\beta(x)}dx\right)^{r+1} \\ &\geq (\frac{m_0}{p^+}-\frac{m_1}{\theta})\|u_n\|^{p^-} - \mu c_3(\frac{B_2}{\alpha^-}-\frac{B_1}{\theta})\|u_n\|^{\alpha^+} \\ &\quad + \lambda c_2\left(\frac{A_1^{r+1}}{\theta(\beta^+)^r} - \frac{1}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\right)\|u_n\|^{\beta^-(r+1)} \end{aligned}$$

dividing the above inequality by $\|u_n\|^{\alpha^+}$, passing to the limit as $n \to +\infty$, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Step 2. We prove that (u_n) has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. From $J'_{\lambda,\mu}(u_n) \to 0$, we have

$$\begin{aligned} J'_{\lambda,\mu}(u_n)(u_n-u) &= M\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n-u)dx \\ &- \lambda\left[\int_\Omega F(x,u_n)dx\right]^r \int_\Omega f(x,u_n)(u_n-u)dx - \mu\int_\Omega g(x,u_n)(u_n-u)dx \to 0. \end{aligned}$$

By the Hölder inequality, we obtain

$$\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| \leq A_2\int_\Omega |u_n|^{\beta(x)-1}|u_n-u|dx \leq c_4\left||u_n|^{\beta(x)-1}\right|_{\frac{\beta(x)}{\beta(x)-1}}|u_n-u|_{\beta(x)}.$$

Since $\beta(x) < p^*(x)$ for all $x \in \overline{\Omega}$, we deduce that $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $L^{\beta(x)}$, hence (u_n) converges strongly to u in $L^{\beta(x)}$, then $\int_\Omega f(x,u_n)(u_n-u)dx \to 0$. By the definition of f and if (u_n) is bounded, we have

$$c_5 \leq \left[A_1\int_\Omega \frac{1}{\beta(x)}|u_n|^{\beta(x)}dx\right]^r \leq \left[\int_\Omega F(x,u_n)dx\right]^r \leq \left[A_2\int_\Omega \frac{1}{\beta(x)}|u_n|^{\beta(x)}dx\right]^r \leq c_6$$

and we obtain

$$\left[\int_\Omega F(x,u_n)dx\right]^r \int_\Omega f(x,u_n)(u_n-u)dx \to 0.$$

Using similar arguments as above, we obtain $\int_\Omega g(x,u_n)(u_n-u)dx \to 0$.

From (M1), we have also $L_{p(x)}(u_n)(u_n-u) = \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\nabla(u_n-u)dx \to 0$. According to the fact that $L_{p(x)}$ satisfies condition S_+, we have $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Hence $J_{\lambda,\mu}$ satisfies the (PS) condition.

In the following, we will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (H2) and (H3) hold.

(H2) For any $u \in Z_k$, $\|u\| = r_k > 1$ (r_k will be given below), we have

$$\begin{aligned}
J_{\lambda,\mu}(u) &= \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} - \mu\int_\Omega G(x,u)dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}\left[\int_\Omega |u|^{\beta(x)}dx\right]^{r+1} - |\mu|\frac{c_7 B_2}{\alpha^-}\|u\|^{\alpha^+} \\
&\geq \begin{cases} \dfrac{m_0}{2p^+}\|u\|^{p^-} - c_8 & \text{if} \quad |u|_{\beta(x)} \leq 1, \\ \dfrac{m_0}{2p^+}\|u\|^{p^-} - \dfrac{\lambda}{r+1}\left(\dfrac{A_2}{\beta^-}\right)^{r+1}(\beta_k\|u\|)^{\beta^+(r+1)} & \text{if} \quad |u|_{\beta(x)} > 1. \end{cases} \\
&\geq \frac{m_0}{p^+}\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}(\beta_k\|u\|)^{\beta^+(r+1)} - c_8.
\end{aligned}$$

Choose $r_k = \left(\frac{2\lambda\beta^+}{m_0}\left(\frac{A_2}{\beta^-}\right)^{r+1}(\beta_k)^{\beta^+(r+1)}\right)^{\frac{1}{p^- - \beta^+(r+1)}}$, we have

$$J_{\lambda,\mu}(u) \geq \frac{m_0}{2}\left(\frac{1}{p^+} - \frac{1}{\beta^+(r+1)}\right)r_k^{p^-} - c_8 \to \infty \quad \text{as} \quad k \to \infty,$$

because of $p^- \leq p^+ < \beta^+(r+1)$, and $\lim_{k\to\infty}\beta_k = 0$.

(H3) Let $u \in Y_k$, such that $\|u\| = \rho_k > r_k > 1$, then

$$\begin{aligned}
J_{\lambda,\mu}(u) &\leq \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} - \mu\int_\Omega G(x,u)dx \\
&\leq \frac{m_1}{p^-}\|u\|^{p^+} - \frac{\lambda}{r+1}\left(\frac{A_1}{\beta^+}\right)^{r+1}\left[\int_\Omega |u|^{\beta(x)}dx\right]^{r+1} + \frac{|\mu|B_2}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx.
\end{aligned}$$

Since $dimY_k < \infty$, all norms are equivalent in Y_k, we obtain

$$J_{\lambda,\mu}(u) \leq \frac{m_1}{p^-}\|u\|^{p^+} - \frac{\lambda}{r+1}\left(\frac{A_1}{\beta^+}\right)^{r+1}\|u\|^{\beta^-(r+1)} + \frac{|\mu|B_2}{\alpha^-}\|u\|^{\alpha^+}.$$

We get that: $J_{\lambda,\mu}(u) \to -\infty$ as $\|u\| \to +\infty$ since $\beta^-(r+1) > p^+ > \alpha^+$. So (H3) holds. From ($f_2$) and ($g_2$) we can deduce that $J_{\lambda,\mu}(u)$ is even. By Lemma 3.1 the proof of (i) is complete.

(ii) We will use Lemma 3.2 to prove conclusion (ii). We prove that there is $k_0 > 0$ such that, for each $k \geq k_0$, there exist $\rho_k > r_k > 0$ such that (L1), (L2) and (L3) are satisfied.

(L1) For any $u \in Z_k$, we have

$$\begin{aligned}
J_{\lambda,\mu}(u) &\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} - \mu\int_\Omega G(x,u)dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^+} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1}c_9\|u\|^{\beta^-(r+1)} - \mu\frac{B_2}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx.
\end{aligned}$$

Notice that $\beta^-(r+1) > p^+$, there exists $\rho_0 > 0$ small enough such that $\frac{\lambda}{r+1}\left(\frac{A_2}{\beta^-}\right)^{r+1} c_9\|u\|^{\beta^-(r+1)} \le \frac{m_0}{2p^+}\|u\|^{p^+}$ as $0 < \rho = \|u\| \le \rho_0$. Then,

$$J_{\lambda,\mu}(u) \ge \begin{cases} \frac{m_0}{2p^+}\|u\|^{p^+} - \mu c_{10}\frac{B_2}{\alpha^-}(\theta_k\|u\|)^{\alpha^-} & \text{if } |u|_{\alpha(x)} \le 1, \\ \frac{m_0}{2p^+}\|u\|^{p^+} - \mu c_{11}\frac{B_2}{\alpha^-}(\theta_k\|u\|)^{\alpha^+} & \text{if } |u|_{\alpha(x)} > 1. \end{cases} \tag{5}$$

Choose

$$\rho_k = \max\left\{ \left(\frac{2p^+B_2}{m_0\alpha^-}\mu c_{10}(\theta_k\|u\|)^{\alpha^-}\right)^{\frac{1}{p^+-\alpha^-}}, \left(\frac{2p^+B_2}{m_0\alpha^-}\mu c_{11}(\theta_k\|u\|)^{\alpha^+}\right)^{\frac{1}{p^+-\alpha^+}} \right\}.$$

Since $p^+ > \alpha^+$, by Proposition 2.7, we have $\rho_k \to 0$ as $k \to \infty$, then $J_{\lambda,\mu}(u) \ge 0$.

(L2) For any $u \in Y_k$, with $\|u\| \le 1$, we have

$$\begin{aligned} J_{\lambda,\mu}(u) &\le \frac{m_1}{p^-}\|u\|^{p^-} - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} - \mu\int_\Omega G(x,u)dx \\ &\le \frac{m_1}{p^-}\|u\|^{p^-} + \lambda c_{12}\|u\|^{\beta^-(r+1)} - \mu c_{13}\|u\|^{\alpha^+}. \end{aligned}$$

Because $\dim Y_k < \infty$, conditions $\beta^-(r+1) > p^- > \alpha^+$, there exists $r_k \in (0, \rho_k)$ such that $J_{\lambda,\mu}(u) < 0$ when $\|u\| = r_k$, i.e. (L2) is satisfied.

(L3) Because $Y_k \cap Z_k \ne \emptyset$, and $r_k < \rho_k$, we have

$$d_k = \inf_{u\in Z_k, \|u\|\le\rho_k} J_{\lambda,\mu}(u) \le b_k = \max_{u\in Y_k, \|u\|=r_k} J_{\lambda,\mu}(u) < 0.$$

For $u \in Z_k$, $\|u\| \le \rho_k$ small enought. From (5), we have

$$J_{\lambda,\mu}(u) \ge \frac{m_0}{2p^+}\|u\|^{p^+} - \mu c_{14}(\theta_k\|u\|)^{\alpha^+} \ge -\mu c_{14}(\theta_k\|u\|)^{\alpha^+}.$$

Since $\theta_k \to 0$ and $\rho_k \to 0$ as $k \to 0$, (L3) holds.

Finally, we verify the $(PS)_c^*$ condition. Suppose that $(u_{n_j}) \subset W_0^{1,p(x)}(\Omega)$ such that $u_{n_j} \in Y_{n_j}$, $J_{\lambda,\mu}(u_{n_j}) \to m$ and $(J_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) \to 0$ as $n_j \to +\infty$.

Similar to proof of **Step 1.** in (i), we can get the boundedness of $\|u_{n_j}\|$. Hence, there exists $u \in W_0^{1,p(x)}(\Omega)$ such that $u_{n_j} \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega) = \overline{\cup_{n_j} Y_{n_j}}$. Then we can find $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. We have

$$\langle J'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\rangle = \langle J'_{\lambda,\mu}(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \langle J'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\rangle.$$

Notice that $u_{n_j} - v_{n_j} \in Y_{n_j}$, it yields

$$\langle J'_{\lambda,\mu}(u_{n_j}), u_{n_j} - u\rangle = \langle (J_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}) - v_{n_j}\rangle + \langle J'_{\lambda,\mu}(u_{n_j}), v_{n_j} - u\rangle \to 0 \text{ as } n \to \infty.$$

Similar to proof of **Step 2.** in (i), we deduce that $u_{n_j} \to u$ in $W_0^{1,p(x)}(\Omega)$. Furthermore, $J'_{\lambda,\mu}(u_{n_j}) \to J'_{\lambda,\mu}(u)$.

Now we claim that u is a critical point of $J_{\lambda,\mu}$. Taking $w_k \in Y_k$, notice that when $n_j \ge k$ we have

$$\langle J'_{\lambda,\mu}(u_{n_j}), w_k\rangle = \langle J'_{\lambda,\mu}(u) - J'_{\lambda,\mu}(u_{n_j}), w_k\rangle + \langle J'_{\lambda,\mu}(u_{n_j}), w_k\rangle$$

$$= \langle J'_{\lambda,\mu}(u) - J'_{\lambda,\mu}(u_{n_j}), w_k\rangle + \langle (J_{\lambda,\mu}|_{Y_{n_j}})'(u_{n_j}), w_k\rangle.$$

Taking $n_j \to \infty$, we obtain $\langle J'_{\lambda,\mu}(u_{n_j}), w_k\rangle = 0$, $\forall w_k \in Y_k$.

So, $J'_{\lambda,\mu}(u_{n_j}) = 0$, this show that $J_{\lambda,\mu}$ satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$.

(iii) By (f_2) and (g_2) we know that $J_{\lambda,\mu}$ is even, next we will prove the two important lemmas for our proof.

Lemma 3.1. *$J_{\lambda,\mu}$ is bounded from below.*

Proof (of Lemma 3.1). From (M1), (f_1) and (g_1) we have

$$J_{\lambda,\mu}(u) = \widehat{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right) - \frac{\lambda}{r+1}\left[\int_\Omega F(x,u)dx\right]^{r+1} - \mu\int_\Omega G(x,u)dx$$

$$\geq \frac{m_0}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \frac{\lambda}{r+1}\left(\frac{A_1}{\beta^+}\right)^{r+1}\left[\int_\Omega |u|^{\beta(x)}dx\right]^{r+1} - \mu\frac{B_2}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx$$

Taking, $\|u\| \geq 1$, we have

$$J_{\lambda,\mu}(u) \geq \frac{m_0}{p^+}\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_1}{\beta^+}\right)^{r+1} c_{15}\|u\|^{\beta^-(r+1)} - \mu\frac{B_2}{\alpha^-}c_{16}\|u\|^{\alpha^+}$$

So $J_{\lambda,\mu}$ is bounded from below, because $\beta^-(r+1) > p^- > \alpha^+$. □

Lemma 3.2. *$J_{\lambda,\mu}$ satisfies the (PS) condition.*

Proof (of Lemma 3.2). Let (u_n) has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$, such that $J_{\lambda,\mu}(u_n) \to c_{17}$ and $J'_{\lambda,\mu}(u_n) \to 0$. Then, by the ceorcivity of $J_{\lambda,\mu}$, the sequence (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$. By the reflexity of $W_0^{1,p(x)}(\Omega)$, for a subsequence still denoted (u_n), such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$. Similar to proof of **Step 2.** in (i), we deduce that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. □

In the sequel, for each $k \in \mathbb{N}$ consider $X_k = \mathrm{span}\{e_1, e_2, e_3, \dots, e_k\}$, the subspace of X. Note that $X_k \hookrightarrow L^{\alpha(x)}(\Omega)$, $X_k \hookrightarrow L^{\beta(x)}(\Omega)$, $1 < \alpha(x) < \beta(x) < p^*(x)$ with continuous immersions. Thus, the norm $W_0^{1,p(x)}(\Omega)$, $L^{\beta(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$ are equivalent on X_k.

Note that using (M1), (f_1) and (g_1), we obtain

$$J_{\lambda,\mu}(u) \leq \frac{m_1}{p^-}\left(\int_\Omega |\nabla u|^{p(x)}\right) - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^+}\right)^{r+1}\left(\int_\Omega |u|^{\beta(x)}\right)^{r+1} - \mu\frac{B_1}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx$$

$$\leq \frac{m_1}{p^-}\|u\|^{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^+}\right)^{r+1} c_{18}\|u\|^{\beta^-(r+1)} - \mu\frac{B_1}{\alpha^+}c_{19}\|u\|^{\alpha^+}$$

if $||u||$ is small enough. Hence,

$$J_{\lambda,\mu}(u) \leq ||u||^{\alpha^+}\left[\left(\frac{m_1}{p^-} - \frac{\lambda}{r+1}\left(\frac{A_2}{\beta^+}\right)^{r+1} c_{18}\right)||u||^{p^- - \alpha^+} - \mu\frac{B_1}{\alpha^+}c_{19}\right].$$

Let R be a positive constant such that

$$\left(\frac{m_1}{p^-}-\frac{\lambda}{r+1}\left(\frac{A_2}{\beta^+}\right)^{r+1}c_{18}\right)R^{p^--\alpha^+}<\mu\frac{B_1}{\alpha^+}c_{19}.$$

Thus, for all $0<r_0<R$, and considering $K=\{u\in X:||u||=r_0\}$, we obtain

$$\begin{aligned}J_{\lambda,\mu}(u)&\leq r_0^{\alpha^+}\left[\left(\frac{m_1}{p^-}-\frac{\lambda}{r+1}\left(\frac{A_2}{\beta^+}\right)^{r+1}c_{18}\right)r_0^{p^--\alpha^+}-\mu\frac{B_1}{\alpha^+}c_{19}\right]\\&<R^{\alpha^+}\left[\left(\frac{m_1}{p^-}-\frac{\lambda}{r+1}\left(\frac{A_2}{\beta^+}\right)^{r+1}c_{18}\right)R^{p^--\alpha^+}-\mu\frac{B_1}{\alpha^+}c_{19}\right]<0=J_{\lambda,\mu}(0).\end{aligned}$$

Which implies $\sup_K J_{\lambda,\mu}(u)<0=J_{\lambda,\mu}(0)$. Because X_k and $\mathbb{R}^k$ are isomorphic and K and S^{k-1} are homeomorphic, we conclude that $\gamma(k)=k$. By the Clark theorem, $J_{\lambda,\mu}$ has at least k pair of different critical points. Because k is arbitrary, we obtain infinitely many critical points of $J_{\lambda,\mu}$.

(iv) When $\lambda<0$, $\mu<0$, we argue by contradiction that $u\in W_0^{1,p(x)}(\Omega)\setminus\{0\}$ is a weak solution of (1). Multiplying (1) by u and integrating by part, we have

$$\begin{aligned}&M\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\int_\Omega|\nabla u|^{p(x)}dx\\&=\lambda\left[\int_\Omega F(x,u)dx\right]^r\int_\Omega f(x,u)udx+\mu\int_\Omega g(x,u)udx.\end{aligned}$$

It is contrary to conditions (M1), (f_1) and (g_1). The proof is complete. □

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